

# Intersection Forms and the Adjunction Formula for Four-manifolds via CR Geometry

Mikhail Chkhenkeli  
Thomas Garrity

Department of Mathematics  
Williams College  
Williamstown, MA 01267  
email: Mikhail.Chkhenkeli@williams.edu  
tgarrity@williams.edu

## Abstract

This is primarily an expository note showing that earlier work of Lai[5] on CR geometry provides a clean interpretation, in terms of a Gauss map, for an adjunction formula for embedded surfaces in an almost complex four manifold. We will see that if  $F$  is a surface with genus  $g$  in an almost complex four-manifold  $M$ , then

$$2 - 2g + F \cdot F - i^*c_1(M) - 2F \cdot C = 0,$$

where  $C$  is a two-cycle on  $M$  pulled back from the cycle of two planes with complex structure in a Grassmannian  $Gr(2, C^N)$  via a Gauss map and where  $i^*c_1(M)$  is the restriction of the first Chern class of  $M$  to  $F$ . The key new term of interest is  $F \cdot C$ , which will capture the points of  $F$  whose tangent planes inherit a complex structure from the almost complex structure of the ambient manifold  $M$ . These complex jump points then determine the genus of smooth representatives of a homology class in  $H_2(M, \mathbb{Z})$ . Further, via polarization, we can use

this formula to determine the intersection form on  $M$  from knowing the nature of the complex jump points of  $M$ 's surfaces.

## 1 The Adjunction Formula

This goal of this note is primarily to point out how a paper of Lai from the early 70s can be used to interpret the intersection form of an almost complex four-manifold. This was earlier pointed out by Eliashberg and Harlamov[3], [2].

Let  $M$  be an almost complex four-manifold. Then at each point  $p$  of  $M$  there is an automorphism  $J : T_pM \rightarrow T_pM$  with  $J^2 = -I$  and such that  $J$  varies smoothly on  $M$ . Thus at each point  $p$  we can identify the tangent plane  $T_pM$  with the complex two-plane  $C^2$ . Let  $F$  be an embedded surface in  $M$ . At most points  $p$  on  $F$ , we expect

$$T_pF \cap JT_pF = 0,$$

meaning that  $T_pF$  will not inherit a complex structure from  $T_pM$ . But there will be some points at which

$$T_pF = JT_pF,$$

namely those points whose real tangent plane inherit the structure of a complex line from  $T_pM$ . We call these points *complex jump points* (In [5], Lai used the term RC-singular point and in [6], Wells used the term nongeneric point). The complex jump points will provide a sharp link between the Euler

characteristic of  $F$ , the self-intersection number of  $F$  and the pullback of the first Chern class of  $M$  to  $F$ . Namely, we will show the adjunction formula

$$2 - 2g + F \cdot F - i^* c_1(M) - 2F \cdot C = 0,$$

where  $i : F \rightarrow M$  is the embedding map and  $F \cdot C$  represents, as we will see, the complex jump points.

We first saw such a formula in a lecture by Kirby and realized that there was a natural proof via the Gauss map. This led us to the fact that the Kirby formula actually was a special case of an extension of an old formula of Lai, who was working in CR geometry. Kirby then pointed out to us the earlier work of Eliashberg and Harlamov, for which we thank him.

The key will be the Gauss map, which is the natural map

$$\sigma : F \rightarrow GR(2, C^N),$$

given via the embedding  $i : T_p F \rightarrow T_p M$  and then choosing enough sections of  $TM$  to have a mapping into a complex affine space. Set

$$C := (\Lambda \in Gr(2, C^N) : J\Lambda = \Lambda),$$

where  $J$  is the automorphism associated with the complex structure on  $C^N$ . The homology of  $C$  can be explicitly computed in terms of the special Schubert cycles of  $Gr(2, C^N)$ . The complex jump points of the surface  $F$  are precisely the pullback of the points  $\sigma(F) \cdot C$ . Lai showed in [5]:

**Theorem 1 (Lai)** *Let  $F$  be a compact real  $k$ -dimensional manifold and  $M$  a real  $2n$ -dimensional almost complex manifold. Let  $i : F \rightarrow M$  be an immersion. Assume  $2n - 2 = k$ . Then*

$$\Omega(F) + \sum_{r=0}^{n-1} \bar{\Omega}(F)^{n-r-1} \cup i^* c_r(M) = 2\sigma^*(\sigma(F) \cdot C).$$

Here  $\Omega(F)$  is the Euler class of  $F$ ,  $\bar{\Omega}(F)$  is the Euler class of the normal bundle of  $F$  in  $M$  and  $\sigma$  and  $C$  are the higher dimensional analogues of our earlier definitions. For real surfaces  $F$  in complex surfaces (and thus real four manifolds)  $M$ , Lai's formula becomes

$$\Omega(F) + \bar{\Omega}(F) - i^* c_1(M) = 2\sigma^*(\sigma(F) \cdot C).$$

But in this case, the Euler characteristic of the normal bundle can be identified with the self-intersection number  $F \cdot F$ . Using that the Euler characteristic of  $F$  can be identified to  $2 - 2g$ , we have at least formally our desired adjunction formula. To actually prove the formula, all we need do is to examine Lai's proof, in which it is assumed that  $M$  is a complex manifold, and then simply to observe that all he needed to use was that  $M$  has the structure of an almost complex manifold. Thus the real purpose of this paper is to point out that Lai's formula, which he no doubt developed without thinking at all about four-manifolds but instead about low codimensional CR structures, can be easily applied to the geometry of four-manifolds.

Note that we have altered the notation of Lai. His  $M$  is our  $F$ , his  $N$  is our  $M$  and his  $DK$  is our  $\sigma(F) \cdot C$ .

## 2 Intersection Forms

The key to the topology of a four-manifold lies in understanding the intersection form on  $H_2(M, \mathbf{Z})$ . The Kirby-Lai formula gives us

$$F \cdot F = 2g - 2 + i^*c_1(M) + 2F \cdot C,$$

and thus expressing the self-intersection number of a surface  $F$  in terms of its genus, the smooth structure of  $M$  and the algebraic number of complex jump points of  $F$ . Once we have the self-intersection number, by polarization we will be able to recover the intersection form, in terms of the geometry of the manifold  $X$ .

Let  $F$  and  $G$  be two elements of  $H_2(M, \mathbf{Z})$ . With only a slight abuse of notation, we can let  $F$  and  $G$  denote smooth representatives of  $F$  and  $G$  and let  $F + G$  denote a smooth representative of the homology class of  $F + G$ . Then we have

$$\begin{aligned} F \cdot G &= \frac{1}{2}(F + G) \cdot (F + G) - \frac{1}{2}F \cdot F - \frac{1}{2}G \cdot G \\ &= \frac{1}{2}(2g_{(F+G)} - 2 + i_{(F+G)}^*c_1(M) + 2(F + G) \cdot C) - \frac{1}{2}(2g_{(F)} - 2 + i_F^*c_1(M) + 2F \cdot C) \\ &\quad - \frac{1}{2}(2g_{(G)} - 2 + i_G^*c_1(M) + 2G \cdot C). \end{aligned}$$

Thus the topology of the intersection form is indeed captured by the geometry of the genus of the three surfaces  $F$ ,  $G$  and  $F + G$ , the first Chern class of  $M$  and the number of complex jump points of  $F$ ,  $G$  and  $F + G$ .

Of course, in practice it is unlikely that one would know this information without already knowing the intersection form. Also, note that we only need the geometric information about smooth representatives for a basis for  $H_2(M, \mathbf{Z})$  and for smooth representatives for pairwise sums of basis elements.

### 3 Bounds on the number of complex jump points on characteristic 2-spheres

In this section we obtain bounds on the number of complex jump points on  $F$  when  $F$  is a 2-sphere smoothly representing a characteristic class of a closed, oriented, simply connected smooth four-manifold  $M$ . By definition, a homology class in  $H_2(M)$  is called characteristic if it is dual to  $w_2(M) \bmod 2$ . In [1] bounds on the self-intersection number of a characteristic 2-sphere are given. By using these bounds and the Kirby - Lai adjunction formula, we obtain the following can be shown.

**Theorem 2** *Let  $i : F \rightarrow M$  be an embedding of a 2-sphere  $F$  representing a characteristic class of a closed, oriented, simply connected smooth four-manifold  $M$ . Let  $n$  be the number of complex jump points on  $F$ . Then*

$$(b^+ - 9b^- + 10 - i^*c_1(M))/2 \leq n \leq (b^+ - b^-/9 - 10/9 - i^*c_1(M))/2,$$

*if  $F \cdot F \leq -1$ ;*

$$(b^+/9 - b^- + 26/9 - i^*c_1(M))/2 \leq n \leq (9b^+ - b^- - 6 - i^*c_1(M))/2,$$

*if  $F \cdot F \geq 1$ .*

Here  $b^+$  and  $b^-$  are the numbers of positive and negative eigenvalues of the intersection form of  $M$ , respectively;  $F \cdot F$  is the self-intersection number of  $F$  and  $i^*c_1(M)$  is the pullback of the first Chern class of  $M$  to  $F$ .

## References

- [1] M. Chkhenkeli, Characteristic 2-Spheres in 4-Manifolds, in preparation, Williams College, 1997.
- [2] Y. Eliashberg, Filling by holomorphic discs and its applications, **Geometry of low-dimensional manifolds**, 2 (Durham, 1989), London Math. Soc. Lecture Note Ser., 151, Cambridge Univ. Press, Cambridge, 1990, pp. 45-67.
- [3] V. M. Harlamov and Y. Eliashberg, On the number of complex points in a complex surface, *Proc. of Leningrad Int. Topology Conference*, 1982, 143-148.
- [4] R. Kirby, An Adjunction Formula for Smooth Surfaces in 4-Manifolds, in preparation.
- [5] H. F. Lai, Characteristic Classes of Real Manifolds Immersed in Complex Manifolds, *Transactions of the American Mathematical Society*, Vol. 172 (1972), 1-33.

- [6] R. O. Wells, Jr., Holomorphic Hulls and Holomorphic Convexity, *Complex Analysis* (Proc. Conf. Rice Univ., Houston, Tex., 1967), Rice Univ. Studies 54 (1968), no. 4, 75-84.